Ellipsoids and elliptic hyperboloids in the Euclidean space $\mathbb{E}^{n+1}$

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Abstract

We establish some characterizations of elliptic hyperboloids (resp., ellipsoids) in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, using the n-dimensional area of the sections cut off by hyperplanes and the $(n+1)$-dimensional volume of regions between parallel hyperplanes. We also give a few characterizations of elliptic paraboloids in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$.

1. Introduction

In what follows we will say that a convex hypersurface of $\mathbb{R}^{n+1}$ is *strictly convex* if the hypersurface is of positive normal curvatures with respect to the unit normal N pointing to the convex side. In particular, the Gauss-Kronecker curvature K is positive with respect to the unit normal N. We will also say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is *strictly convex* if the graph of f is strictly convex with respect to the upward unit normal N.

Consider a smooth function $g : \mathbb{R}^{n+1} \to \mathbb{R}$. We denote by R_g the set of all regular values of the function g. We assume that there exists an interval $S_g \subset R_g$ such that for every $k \in S_g$, the level hypersurface $M_k = g^{-1}(k)$ is a smooth strictly convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. We also denote by S_g the maximal interval in R_g which satisfies the above property.

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If $k \in S_g$, then we may choose a maximal interval $I_k \subset S_g$ so that each M_{k+h} with $k+h \in I_k$ lies in the convex side of M_k. Note that I_k is of the form (k, a) with $a > k$ or (b, k) with $b < k$ according as the gradient ∇g of the function g points to the convex side of M_k or not.

For examples, consider two functions $g_{\pm} : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $g(x, z) = z^2 \pm (a_1^2 x_1^2 + \cdots + a_n^2 x_n^2)$ with positive constants $a_1, \cdots, a_n$. Then, for the function g_- we have $R_{g_-} = R - \{0\}$, $S_{g_-} = (0, \infty)$ and $I_k = (k, \infty)$, $k \in S_{g_-}$. For g_+, we get $R_{g_+} = S_{g_+} = (0, \infty)$ and $I_k = (0, k)$ with $k \in S_{g_+}$.

For a fixed point $p \in M_k$ with $k \in S_g$ and a sufficiently small h with $k + h \in I_k$, we consider the tangent hyperplane Φ of M_{k+h} at some point $v \in M_{k+h}$, which is parallel to the tangent hyperplane Ψ of M_k at $p \in M_k$. We denote by $A_p^*(k, h), V_p^*(k, h)$ and $S_p^*(k, h)$ the n-dimensional area of the section in Φ enclosed by $\Phi \cap M_k$, the $(n + 1)$-dimensional volume of the region bounded by M_k and the hyperplane Φ, and the n-dimensional surface area of the region of M_k between the two hyperplanes Φ and Ψ, respectively.

In [3], the author and Y. H. Kim studied hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of some function $f : \mathbb{R}^n \to \mathbb{R}$. In our notations, they proved the following characterization theorem for elliptic paraboloids in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, which extends a result in [2].

Proposition 1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly convex function. We consider the function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $g(x, z) = z - f(x), x = (x_1, \cdots, x_n)$. Then, the following are equivalent.

1) For a fixed $k \in R$, $V_p^*(k, h)$ is a nonnegative function $\phi(h)$, which depends only on h.
2) For a fixed $k \in R$, $A_p^*(k, h)/|\nabla g(p)|$ is a nonnegative function $\psi(h)$, which depends only on h. Here ∇g denotes the gradient of g.
3) The function $f(x)$ is a quadratic polynomial given by $f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2$ with $a_i > 0, i = 1, 2, \cdots, n$, and hence every level hypersurface M_k of g is an elliptic paraboloid.

Note that in the above proposition, $R_g = S_g = R$ and $I_k = (k, \infty)$.

In particular, when $n = 2$, in a long series of propositions, Archimedes proved that every level surface M_k (paraboloid of rotation) of the function $g(x, y, z) = z - a^2(x^2 + y^2)$ in the 3-dimensional Euclidean space $\mathbb{E}^3$ satisfies $V_p^*(k, h) = ch^2$ for some constant c ([5], p.66 and Appendix A and B).

In this paper, we study the family of strictly convex level hypersurfaces $M_k, k \in S_g$ of a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies the following conditions.

(V^*): For $k \in S_g$ with $k + h \in I_k$, $V_p^*(k, h)$ with $p \in M_k$ is a nonnegative function $\phi_k(h)$, which depends only on k and h.

(A^*): For $k \in S_g$ with $k+h \in I_k$, $A_p^*(k, h)/|\nabla g(p)|$ with $p \in M_k$ is a nonnegative function $\psi_k(h)$, which depends only on k and h.

(S^*): For $k \in S_g$ with $k+h \in I_k$, $S_p^*(k, h)/|\nabla g(p)|$ with $p \in M_k$ is a nonnegative function $\eta_k(h)$, which depends only on k and h.

As a result, first of all, we establish the following characterizations of elliptic hyperboloids.

Theorem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$ with $\alpha \neq 1$, let's denote by g the function defined by $g(x, z) = z^\alpha - f(x)$. Suppose that the level hypersurfaces $M_k (k \in S_g)$ of g in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex. Then the following are equivalent.
1) The function g satisfies Condition (V^*).
2) The function g satisfies Condition (A^*).
3) For $k \in S_g$, $K(p)|\nabla g(p)|^{n+2} = c(k)$ is constant on M_k, where $K(p)$ denotes the Gauss-Kronecker curvature of M_k at $p \in M_k$ with respect to the unit normal pointing to the convex side.
4) The function g is given by

$$g(x, z) = z^2 - (a_1^2 x_1^2 + \cdots + a_n^2 x_n^2),$$

where $a_i > 0, i = 1, 2, \cdots, n$. In this case, $R_g = R - \{0\}$, $S_g = (0, \infty)$ and $I_k = (k, \infty)$, $k \in S_g$.

Next, in the similar way to the proof of Theorem 2, we prove the following characterizations of ellipsoids.

Theorem 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$ with $\alpha \neq 1$, let's denote by g the function defined by $g(x, z) = z^\alpha + f(x)$. Suppose that the level hypersurfaces $M_k (k \in S_g)$ of g in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex. Then the following are equivalent.
1) The function g satisfies Condition (V^*).
2) The function g satisfies Condition (A^*).

3) For $k \in S_g$, $K(p)|\nabla g(p)|^{n+2} = c(k)$ is constant on M_k, where $K(p)$ denotes the Gauss-Kronecker curvature of M_k at $p \in M_k$ with respect to the unit normal pointing to the convex side.
4) The function g is given by

$$g(x,z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2,$$

where $a_i > 0, i = 1, 2, \cdots, n$. In this case, $R_g = S_g = (0, \infty)$ and $I_k = (0, k)$, $k \in S_g$.

In view of the above theorems and Lemma 9 in Section 2, it is natural to ask the following question.

Question 4. Which functions $g : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy Condition (S^*)?

Partially, we answer Question 4 as follows.

Theorem 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$ with $\alpha \neq 1$, let's denote by g the function defined by $g(x,z) = z^\alpha - f(x)$. Suppose that the level hypersurfaces $M_k (k \in S_g)$ of g in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex. Then, the function g does not satisfy Condition (S^*).

In [3], using harmonic function theory, the author and Y. H. Kim proved Theorem 5 when $\alpha = 1$.

Finally, we generalize the characterization theorem of [3] for elliptic paraboloids in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ as follows.

Theorem 6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$ with $\alpha \neq 2$, let's denote by g the function defined by $g(x,z) = z^\alpha - f(x)$. Suppose that the level hypersurfaces $M_k (k \in S_g)$ of g in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex. Then the following are equivalent.

1)The function g satisfies Condition (V^*).
2) The function g satisfies Condition (A^*).
3) For $k \in S_g$, $K(p)|\nabla g(p)|^{n+2} = c(k)$ is constant on M_k.
4) The function g is given by

$$g(x,z) = z - a_1^2 x_1^2 - \cdots - a_n^2 x_n^2,$$

where $a_1, \cdots, a_n$ are positive constants. In this case, $R_g = S_g = R$ and $I_k = (k, \infty)$.

Throughout this article, all objects are smooth and connected, unless otherwise mentioned.

2. Preliminaries

Suppose that M is a smooth strictly convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the unit normal N pointing to the convex side. For a fixed point $p \in M$ and for a sufficiently small $t > 0$, consider the hyperplane Φ passing through the point $p + tN(p)$ which is parallel to the tangent hyperplane Ψ of M at p.

We denote by $A_p(t), V_p(t)$ and $S_p(t)$ the n-dimensional area of the section in Φ enclosed by $\Phi \cap M$, the $(n+1)$-dimensional volume of the region bounded by the hypersurface and the hyperplane Φ and the n-dimensional surface area of the region of M between the two hyperplanes Φ and Ψ, respectively.

Now, we may introduce a coordinate system $(x, z) = (x_1, x_2, \cdots, x_n, z)$ of $\mathbb{E}^{n+1}$ with the origin p, the tangent hyperplane of M at p is the hyperplane $z = 0$. Furthermore, we may assume that M is locally the graph of a non-negative strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$. Hence N is the unit normal pointing upward.

Then, for a sufficiently small $t > 0$ we have

$$A_p(t) = \int_{f(x)<t} 1dx, \tag{2.1}$$

$$V_p(t) = \int_{f(x)<t} \{t - f(x)\}dx \tag{2.2}$$

and

$$S_p(t) = \int_{f(x)<t} \sqrt{1 + |\nabla f|^2}dx, \tag{2.3}$$

where $x = (x_1, x_2, \cdots, x_n)$, $dx = dx_1 dx_2 \cdots dx_n$ and ∇f denote the gradient vector of the function f.

Note that we also have

$$\begin{aligned} V_p(t) &= \int_{f(x)<t} \{t - f(x)\}dx \\ &= \int_{z=0}^{t} \{\int_{f(x)<z} 1dx\}dz. \end{aligned} \tag{2.4}$$

Hence, together with the fundamental theorem of calculus, (2.4) shows that

$$V_p'(t) = \int_{f(x)<t} 1dx = A_p(t). \tag{2.5}$$

In order to prove our theorems, first of all, we need the following.

Lemma 7. Suppose that the Gauss-Kronecker curvature $K(p)$ of M at p is positive with respect to the unit normal N pointing to the convex side of M. Then we have the following.

1)

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^n} A_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \tag{2.6}$$

2)

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^{n+2}} V_p(t) = \frac{(\sqrt{2})^{n+2} \omega_n}{(n+2)\sqrt{K(p)}}, \tag{2.7}$$

3)

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^n} S_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \tag{2.8}$$

where ω_n denotes the volume of the n-dimensional unit ball.

Proof. For proofs of 1), 2) and 3) with n=2, see Lemma 7 of [2]. For a proof of 3) with arbitrary n, see Lemma 8 of [3].

Now, we prove the following.

Lemma 8. Consider the family of strictly convex level hypersurfaces $M_k = g^{-1}(k)$ of a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies Condition (V^*). Then, for each $k \in S_g$, on M_k we have

$$K(p)|\nabla g(p)|^{n+2} = c(k), \tag{2.9}$$

which is independent of $p \in M_k$, where $K(p)$ is the Gauss-Kronecker curvature of M_k at p with respect to the unit normal N pointing to the convex side and $\nabla g(p)$ denotes the gradient of g at p.

Proof. By considering $-g$ if necessary, we may assume that I_k is of the form $[k, a]$ with $a > k$, that is, $N = \nabla g/|\nabla g|$ on M_k. For a fixed point $p \in M_k$ and a small $t > 0$, we have

$$V_p(t) = V_p^*(k, h(t)) = \phi_k(h(t)),$$

where $h = h(t)$ is a positive function with $h(0) = 0$. By differentiating with respect to t, we get

$$A_p(t) = V_p'(t) = \phi_k'(h)h'(t), \tag{2.10}$$

where $\phi_k'(h)$ denotes the derivative of ϕ_k with respect to h. Hence we obtain

$$\frac{1}{(\sqrt{t})^n}A_p(t) = \frac{\phi_k'(h)}{(\sqrt{h})^n}(\sqrt{\frac{h(t)}{t}})^n h'(t). \tag{2.11}$$

Now we claim that

$$\lim_{t\to 0} h'(t) = |\nabla g(p)|. \tag{2.12}$$

Assuming (2.12), we also get

$$\lim_{t\to 0} \sqrt{\frac{h(t)}{t}} = \sqrt{|\nabla g(p)|}. \tag{2.13}$$

Let us put $\lim_{h\to 0} \phi_k'(h)/(\sqrt{h})^n = \gamma(k)$, which is independent of p. Then it follows from (2.11), (2.12), (2.13) and Lemma 7 that

$$K(p)|\nabla g(p)|^{n+2} = \frac{2^n\omega_n^2}{\gamma(k)^2}, \tag{2.14}$$

which is constant on the level hypersurface M_k. Thus it suffices to show that (2.12) holds.

In order to prove (2.12), we consider an orthonormal basis $E_1, \cdots, E_n, N(p)$ of $\mathbb{E}^{n+1}$ at $p \in M_k$, where $E_1, \cdots, E_n \in T_p(M_k)$ and $N(p) = \nabla g(p)/|\nabla g(p)|$ is the unit normal pointing to the convex side. We consider a 1-parameter family Φ_t of hyperplanes $\Phi_t(s_1, \cdots, s_n) = p + tN(p) + \Sigma_{i=1}^n s_i E_i$, which are parallel to the tangent hyperplane Φ_0 of M_k at p. For small $t > 0$, there exist $s_i = s_i(t), i = 1, 2, \cdots, n$ with $s_i(0) = 0$ such that Φ_t is tangent to $M_{k+h(t)}$ at $s_i = s_i(t), i = 1, 2, \cdots, n$. Hence we have

$$k + h(t) = g(\Phi_t(s_1, \cdots, s_n)) = g(p + tN(p) + \Sigma_{i=1}^n s_i(t)E_i). \tag{2.15}$$

Thus, by differentiating with respect to t, we get

$$\frac{dh}{dt} = \left\langle \nabla g(\Phi_t(s_1, \cdots, s_n)), N(p) + \Sigma_{i=1}^n \frac{ds_i}{dt}E_i \right\rangle. \tag{2.16}$$

Note that $\Phi_t(s_1, \cdots, s_n) \to p$ as t tends to 0. Therefore, we obtain

$$\lim_{t\to 0} \frac{dh}{dt} = \lim_{t\to 0} \langle \nabla g(p), N(p)\rangle = |\nabla g(p)|, \tag{2.17}$$

which completes the proof of (2.12). This completes the proof. □

Similarly to the proof of Lemma 8, we may obtain

Lemma 9. Consider the family of strictly convex level hypersurfaces $M_k = g^{-1}(k)$ of a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies either Condition (A^*) or Condition (S^*). Then, on M_k with $k \in S_g$ we have

$$K(p)|\nabla g(p)|^{n+2} = d(k), \tag{2.18}$$

which is independent of $p \in M_k$, where $K(p)$ is the Gauss-Kronecker curvature of M_k at p with respect to the unit normal N pointing to the convex side and $\nabla g(p)$ denotes the gradient of g at p.

Proof. As in the proof of Lemma 8, we may assume that I_k is of the form $[k, a]$ with $a > k$, that is, $N = \nabla g/|\nabla g|$ on M_k. For a fixed point $p \in M_k$ and a small $t > 0$, we have $A_p(t) = A_p^*(k, h(t))$ for some positive function $h = h(t)$ with $h(0) = 0$. Suppose that M_k satisfies Condition (A^*). Then, we have

$$A_p(t) = A_p^*(k, h(t)) = \psi_k(h(t))|\nabla g(p)|. \tag{2.19}$$

Hence we obtain

$$\frac{1}{(\sqrt{t})^n} A_p(t) = \frac{\psi_k(h)}{(\sqrt{h})^n} (\sqrt{\frac{h(t)}{t}})^n |\nabla g(p)|. \tag{2.20}$$

We put $\lim_{h \to 0} \psi_k(h)/(\sqrt{h})^n = \beta(k)$, which is independent of $p \in M_k$. Then it follows from (2.13), (2.20) and Lemma 7 that

$$K(p)|\nabla g(p)|^{n+2} = \frac{2^n \omega_n^2}{\beta(k)^2}, \tag{2.21}$$

which is constant on the level hypersurface M_k.

The remaining case can be treated similarly. This completes the proof. $\square$

3. Ellipsoids and elliptic hyperboloids

In this section, first of all, we prove Theorem 2.

For a nonzero real number α with $\alpha \neq 1$ and a nonnegative convex function $f(x)$ defined on $\mathbb{R}^n$, we consider the function $g(x, z) = z^\alpha - f(x)$. We assume that the level hypersurfaces $M_k, k \in S_g$ defined by $g(x, z) = k$ are all strictly convex, and hence each $M_k, k \in S_g$ has positive Gauss-Kronecker curvature K with respect to the unit normal N pointing to the convex side.

On each M_k, by differentiating, we have for a fixed point $p = (x, z) \in M_k$,

$$\begin{aligned}
&\nabla f = \alpha z^{\alpha-1} \nabla z, \\
&|\nabla g(p)|^2 = \alpha^2 z^{2\alpha-2} + |\nabla f(x)|^2, \\
&z_{ij} = \frac{1}{\alpha^2 z^{2\alpha-1}} (\alpha z^\alpha f_{ij} - (\alpha - 1) f_i f_j), \quad i, j = 1, 2, \cdots, n,
\end{aligned} \tag{3.1}$$

where z_i denotes the partial derivative of z with respect to $x_i, i = 1, 2, \cdots, n$, and so on. The Gauss-Kronecker curvature $K(p)$ of M_k at p is given by ([6])

$$\begin{aligned} K(p) &= \frac{\det(z_{ij})}{(\sqrt{1+|\nabla z|^2})^{n+2}} \\ &= \frac{\alpha^{n+2} z^{(\alpha-1)(n+2)} \det(z_{ij})}{(\sqrt{\alpha^2 z^{2\alpha-2} + |\nabla f(x)|^2})^{n+2}}, \end{aligned} \tag{3.2}$$

where the second equality follows from (3.1). Thus, it follows from (3.1) and (3.2) that

$$\begin{aligned} K(p)|\nabla g(p)|^{n+2} &= \alpha^{n+2} z^{(\alpha-1)(n+2)} \det(z_{ij}) \\ &= \frac{1}{\alpha^{n-2} z^{\alpha n-2\alpha+2}} \det(\alpha z^\alpha f_{ij} - (\alpha-1) f_i f_j). \end{aligned} \tag{3.3}$$

First, suppose that the function $g(x, z) = z^\alpha - f(x)$ satisfies Condition (V^*) or Condition (A^*). Then, it follows from Lemma 8 or Lemma 9 that the function g satisfies the condition 3) of Theorem 2. That is, there exists a constant $c(k)$ depending on k such that

$$\det(\alpha z^\alpha f_{ij} - (\alpha-1) f_i f_j) = \alpha^{n-2} c(k) z^{\alpha n - 2\alpha + 2}. \tag{3.4}$$

By substituting $z^\alpha = f(x) + k$ into (3.4), we see that $f(x)$ satisfies

$$\det(\alpha(f(x)+k) f_{ij} - (\alpha-1) f_i f_j) = \alpha^{n-2} c(k) (f(x)+k)^{n-2+2/\alpha}. \tag{3.5}$$

We denote by $A_i, i = 1, 2, \cdots, n$ the i-th column vector of the matrix in the left hand side of (3.5). Then we have

$$A_i = \alpha(f(x)+k) B_i - C_i, \tag{3.6}$$

where

$$B_i = \begin{pmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{in} \end{pmatrix} = \nabla f_i, \quad C_i = (\alpha-1) f_i \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = (\alpha-1) f_i \nabla f. \tag{3.7}$$

Hence, it follows from the multilinear alternating property of determinant function that

$$\begin{aligned} \det(A_1, \cdots, A_n) &= \alpha^n (f(x)+k)^n \det(B_1, \cdots, B_n) \\ &\quad - \alpha^{n-1} (f(x)+k)^{n-1} \{\det(C_1, B_2, \cdots, B_n) \\ &\quad + \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n)\}. \end{aligned} \tag{3.8}$$

Since $\det(f_{ij}) = \det(B_1, \cdots, B_n)$, it follows from (3.5) and (3.8) that

$$\begin{aligned} c(k)(f(x)+k)^{(2-\alpha)/\alpha} &= \alpha^2(f(x)+k)\det(f_{ij}) - \alpha\{\det(C_1, B_2, \cdots, B_n) \\ &+ \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n)\} \\ &= A(x)k + B(x), \end{aligned} \tag{3.9}$$

where we use the following notations.

$$\begin{aligned} A(x) &= \alpha^2 \det(f_{ij}), \\ B(x) &= \alpha^2 f(x)\det(f_{ij}) - \alpha\{\det(C_1, B_2, \cdots, B_n) \\ &+ \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n)\}. \end{aligned} \tag{3.10}$$

Note that the right hand side of (3.9) is a linear polynomial in k with functions in $x = (x_1, \cdots, x_n)$ as coefficients. Furthermore, note that for each k, $c(k)$ is positive and $f(x)+k$ is a nonconstant function in x. It follows from (3.9) that

$$c(k)^\alpha = (A(x)k + B(x))^\alpha (k + f(x))^{\alpha-2}. \tag{3.11}$$

Suppose that α is a nonzero real number with $\alpha \neq 1, 2$. Then, by using logarithmic differentiation of (3.11) with respect to $x_i, i = 1, 2, \cdots, n$, we get

$$\alpha(\nabla A(x)k + \nabla B(x))(k + f(x)) + (\alpha - 2)(A(x)k + B(x))\nabla f(x) = 0, \tag{3.12}$$

which is a quadratic polynomial in k. It follows from (3.12) and the assumption $\alpha \neq 0, 1, 2$ that $\nabla f(x) = 0$, which is a contradiction.

Thus, by assumption, we see that $\alpha = 2$ is the only possible case. In this case, (3.9) implies that for some constants a and b, $c(k) = ak + b$ with $A(x) = a$ and $B(x) = b$. It follows from (3.10) that

$$\det(f_{ij}) = \frac{a}{4}, \tag{3.13}$$

and

$$\det(C_1, B_2, \cdots, B_n) + \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n) = \frac{1}{2}(af(x) - b), \tag{3.14}$$

where

$$B_i = \nabla f_i, \quad C_i = f_i \nabla f, \quad i = 1, 2, \cdots, n. \tag{3.15}$$

Since $f(x)$ is a nonnegative strictly convex function, (3.13) shows that $\det(f_{ij})$ is a positive constant on $\mathbb{R}^n$. Hence $f(x)$ is a quadratic polynomial given by ([1], [4])

$$f(x_1, \cdots, x_n) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, \quad a_1, \cdots, a_n > 0. \tag{3.16}$$

Thus, the level hypersurfaces must be the elliptic hyperboloids $M_k = g^{-1}(k)$, where $g(x,z) = z^2 - (a_1^2 x_1^2 + \cdots + a_n^2 x_n^2), z > 0$ with $k > 0$ and $a_1, \cdots, a_n > 0$.

Conversely, consider the function g given by $g(x,z) = z^2 - f(x), z > 0$ with $k > 0$, where $f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, a_1, \cdots, a_n > 0$. For the function g, we have $R_g = R - \{0\}$, $S_g = (0, \infty)$ and $I_k = (k, \infty)$, $k \in S_g$.

For a fixed $k > 0$ and a small $h > 0$, consider the tangent hyperplane Ψ of M_k at a point $p \in M_k$. There exists a point $v \in M_{k+h}$ such that the tangent hyperplane Φ of M_{k+h} at v is parallel to the hyperplane Ψ. The two points p and v of tangency are related by

$$v = \frac{\sqrt{k+h}}{\sqrt{k}} p, \quad p = (p_1, \cdots, p_n, \sqrt{r^2 + k}), r^2 = a_1^2 p_1^2 + \cdots + a_n^2 p_n^2. \tag{3.17}$$

Note that $V_p^*(k,h)$ denote the $(n+1)$-dimensional volume of the region of M_k cut off by the hyperplane Φ.

Then the linear mapping

$$T_1(x_1, x_2, \cdots, x_n, z) = (a_1 x_1, a_2 x_2, \cdots, a_n x_n, z) \tag{3.18}$$

transforms M_k (resp., M_{k+h}) onto a hyperboloid of revolution $M'_k : z^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + k$, (resp., $M'_{k+h} : z^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + k + h$), Φ to a hyperplane Φ', $p \in M_k$ and $v \in M_{k+h}$ to points of tangency $p' = (p'_1, \cdots, p'_n, \sqrt{(r')^2 + k}) \in M'_k$ and $v' = (\sqrt{k+h}/\sqrt{k})p' \in M'_{k+h}$, respectively, where $(r')^2 = \Sigma(p'_i)^2$. If we let $V'^*_{p'}(k,h)$ denote the volume of the region of M'_k cut off by the hyperplane Φ', then we get

$$V'^*_{p'}(k,h) = a_1 \cdots a_n V_p^*(k,h). \tag{3.19}$$

Let's consider the rotation A around the z-axis which maps the point p' of tangency to $p'' = (0, \cdots, 0, r', \sqrt{(r')^2 + k})$. Then the rotation A takes v' to $v'' = (\sqrt{k+h}/\sqrt{k})p''$. Note that the 1-parameter group $B(t)$ on the $x_n z$-plane defined by

$$B(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \tag{3.20}$$

takes the upper hyperbola $z^2 = x_n^2 + k, z > 0$ (resp., $z^2 = x_n^2 + k + h, z > 0$) onto itself. Hence, there exists a parameter t_0 such that $B(t_0)$ maps p'' to $p''' = (0, \ldots, 0, \sqrt{k})$ (resp., v'' to $v''' = (0, \ldots, 0, \sqrt{k+h})$).

We consider the linear mapping $T_2 = \bar{B}(t_0) \circ A$ of $\mathbb{R}^{n+1}$, where $\bar{B}(t_0)$ denotes the extended linear mapping of $B(t_0)$ on $\mathbb{R}^{n+1}$ fixing $x_1 \cdots x_{n-1}$-plane. Then the linear mapping T_2 takes

the hyperboloid of rotation M'_k (resp., M'_{k+h}) onto itself, p' and v' to the points of tangency $p''' = (0, \ldots, 0, \sqrt{k})$ and $v''' = (0, \ldots, 0, \sqrt{k+h})$, Φ' to the hyperplane $\Phi'' : z = \sqrt{k+h}$.

Due to the volume-preserving property of T_2, we obtain

$$V'^{*}_{p'}(k, h) = V^*(k, h), \tag{3.21}$$

where $V^*(k, h)$ denotes the volume of the region of M'_k cut off by the hyperplane Φ''. Together with (3.19), it follows from (3.21) that

$$V^*_p(k, h) = \frac{\omega_n}{a_1 \cdots a_n}\{\sqrt{k+h}h^{n/2} - n\int_0^{\sqrt{h}} \sqrt{r^2+k}r^{n-1}dr\}, \tag{3.22}$$

where ω_n denotes the volume of the n-dimensional unit ball. Hence, we see that $V^*_p(k, h)$ is independent of the point $p \in M_k$, which is denoted by $\phi_k(h)$. Thus the function g given by $g(x, z) = z^2 - f(x), z > 0$ satisfies Condition (V^*).

Finally, we show that the function g given by $g(x, z) = z^2 - f(x), z > 0$ with $k > 0$, where $f(x) = a_1^2x_1^2 + \cdots + a_n^2x_n^2, a_1, \cdots, a_n > 0$ satisfies Condition (A^*). For a fixed point $p = (p_1, \cdots, p_n, \sqrt{r^2+k}) \in M_k$, where $r^2 = a_1^2p_1^2 + \cdots + a_n^2p_n^2$, and a small $t \in R$, we have $V_p(t) = \phi_k(h(t))$ for some $h = h(t)$ with $h(0) = 0$. By differentiating with respect to t, from (2.5) we get

$$A_p(t) = V'_p(t) = \phi'_k(h)h'(t), \tag{3.23}$$

where $\phi'_k(h)$ denotes the derivative of ϕ_k with respect to h.

With the aid of (3.17), it is straightforward to show that

$$t = \frac{2\sqrt{k}(\sqrt{k+h} - \sqrt{k})}{|\nabla g(p)|}. \tag{3.24}$$

Hence we get

$$h(t) = \frac{1}{4k}|\nabla g(p)|^2t^2 + |\nabla g(p)|t. \tag{3.25}$$

It follows from (3.24), (3.25) and (3.23) that

$$A^*_p(k, h) = A_p(t) = \frac{\sqrt{k+h}}{\sqrt{k}}\phi'_k(h)|\nabla g(p)|, \tag{3.26}$$

which shows that the function g given by $g(x, z) = z^2 - f(x), z > 0$ satisfies Condition (A^*).

It follows from (3.9) and (3.10) with $\alpha = 2$ that the function g given by $g(x, z) = z^2 - f(x), z > 0$ satisfies

$$K(p)|\nabla g(p)|^{n+2} = c(k) = 2^{n+2}a_1^2a_2^2 \cdots a_n^2k. \tag{3.27}$$

This completes the proof of Theorem 2.

Next, we prove Theorem 3 as follows.

For a nonzero real number α with $\alpha \neq 1$ and a nonnegative convex function $f(x)$ defined on $\mathbb{R}^n$, we consider the function $g(x,z) = z^\alpha + f(x)$. We assume that the level hypersurfaces $M_k, k \in S_g$ defined by $g(x,z) = k$ are all strictly convex, and hence each $M_k, k \in S_g$ has positive Gauss-Kronecker curvature K with respect to the unit normal N pointing to the convex side.

Suppose that the function g satisfies Condition (V^*) or Condition (A^*). Then, it follows from Lemma 8 or Lemma 9 that M_k satisfies the condition 3) of Theorem 3. Then, changing $f(x)$ by $-f(x)$ in the proof of Theorem 2, (3.11) shows that $f(x)$ satisfies

$$(-1)^{\alpha n} c(k)^\alpha = (A(x)k + B(x))^\alpha (k - f(x))^{\alpha-2}, \tag{3.28}$$

where

$$\begin{aligned} A(x) &= \alpha^2 \det(f_{ij}), \\ B(x) &= -\alpha^2 f(x) \det(f_{ij}) + \alpha\{\det(C_1, B_2, \cdots, B_n) \\ &\quad + \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n)\}, \end{aligned} \tag{3.29}$$

and $B_i = \nabla f_i, C_i = (\alpha - 1) f_i \nabla f, i = 1, 2, \cdots, n$. Since $c(k) > 0$, the logarithmic differentiation of (3.28) shows that $\alpha = 2$, and hence for some constants a and b, we get $(-1)^n c(k) = ak+b$ with $\det(f_{ij}) = a/4$. Since $f(x)$ is a nonnegative strictly convex function, this implies that $\det(f_{ij})$ is a positive constant on $\mathbb{R}^n$. By the same argument as in the proof of Theorem 2, we see that $f(x)$ is a quadratic polynomial given by

$$f(x_1, \cdots, x_n) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, \quad a_1, \cdots, a_n > 0. \tag{3.30}$$

Thus, the level hypersurfaces must be the ellipsoids given by $g(x,z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2 = k$ with $k > 0$ and $a_1, \cdots, a_n > 0$.

Conversely, we consider the function g given by $g(x,z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2$ with $a_1, \cdots, a_n > 0$. For the function g, we have $R_g = S_g = (0, \infty)$ and $I_k = (0, k)$, $k \in S_g$.

For a fixed $k > 0$ and a point $p \in M_k$, consider the tangent hyperplane Ψ of M_k at p. For a sufficiently small $h < 0$ with $k + h > 0$, there exists a point $v \in M_{k+h}$ such that the tangent hyperplane Φ of M_{k+h} at v is parallel to the hyperplane Ψ. The two points p and v of tangency are related by

$$v = \frac{\sqrt{k+h}}{\sqrt{k}} p, \quad p = (p_1, \cdots, p_n, \sqrt{k - r^2}), r^2 = a_1^2 p_1^2 + \cdots + a_n^2 p_n^2. \tag{3.31}$$

Then the linear mapping

$$T_1(x_1, x_2, \cdots, x_n, z) = (a_1x_1, a_2x_2, \cdots, a_nx_n, z) \tag{3.32}$$

transforms M_k (resp., M_{k+h}) onto a hypersphere $M'_k : x_1^2 + x_2^2 + \cdots + x_n^2 + z^2 = k$, (resp., $M'_{k+h} : x_1^2 + x_2^2 + \cdots + x_n^2 + z^2 = k + h$), Φ to a hyperplane Φ', $p \in M_k$ and $v \in M_{k+h}$ to points of tangency $p' = (p'_1, \cdots, p'_n, \sqrt{k - (r')^2}) \in M'_k$ and $v' = (\sqrt{k+h}/\sqrt{k})p' \in M'_{k+h}$, respectively, where $(r')^2 = \Sigma(p'_i)^2$. The corresponding volume $V'^*_{p'}(k, h)$ is given by

$$V'^*_{p'}(k, h) = a_1 \cdots a_n V^*_p(k, h). \tag{3.33}$$

By the symmetry of hyperspheres M'_k and M'_{k+h} centered at the origin, we see that $V'^*_{p'}(k, h)$ is independent of the point p'. This, together with (3.33), shows that the function g satisfies Condition (V^*).

Finally, we show that the function g given by $g(x, z) = z^2 + a_1^2x_1^2 + \cdots + a_n^2x_n^2$ with $a_1, \cdots, a_n > 0$ satisfies Condition (A^*).

For a fixed point $p = (p_1, \cdots, p_n, \sqrt{k - r^2}) \in M_k$, where $r^2 = a_1^2p_1^2 + \cdots + a_n^2p_n^2$, and a small $t > 0$, we have $V_p(t) = \phi_k(h(t))$ for some negative function $h = h(t)$ with $h(0) = 0$. By differentiating with respect to t, we get from (2.5)

$$A_p(t) = V'_p(t) = \phi'_k(h)h'(t), \tag{3.34}$$

where $\phi'_k(h)$ denotes the derivative of ϕ_k with respect to h.

With the help of (3.31), it is straightforward to show that the distance t from $p \in M_k$ to the tangent hyperplane Φ to M_{k+h} at $v \in M_{k+h}$ is given by

$$t = \frac{2\sqrt{k}(\sqrt{k} - \sqrt{k+h})}{|\nabla g(p)|}. \tag{3.35}$$

Hence we get

$$h(t) = \frac{1}{4k}|\nabla g(p)|^2t^2 - |\nabla g(p)|t. \tag{3.36}$$

It follows from (3.34), (3.35) and (3.36) that

$$A^*_p(k, h) = A_p(t) = -\frac{\sqrt{k+h}}{\sqrt{k}}\phi'_k(h)|\nabla g(p)|. \tag{3.37}$$

This shows that the function g given by $g(x, z) = z^2 + a_1^2x_1^2 + \cdots + a_n^2x_n^2$ with $a_1, \cdots, a_n > 0$ satisfies Condition (A^*).

It follows from (3.28) and (3.29) with $\alpha = 2$ that the family M_k of ellipsoids satisfies

$$K(p)|\nabla g(p)|^{n+2} = c(k) = 2^{n+2}a_1^2a_2^2 \cdots a_n^2 k. \tag{3.38}$$

This completes the proof of Theorem 3.

4. Condition (S^*)

In this section, we prove Theorem 5.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a real number $\alpha \in R$ with $\alpha \neq 0, 1$, we consider the function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $g(x, z) = z^\alpha - f(x)$.

Suppose that the level hypersurfaces $M_k, k \in S_g$ of g in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex and that the function g satisfies Condition (S^*). Then, as in the proof of Theorem 2, we can show that the function g is given by $g(x, z) = z^2 - f(x), z > 0$ with $k > 0$, where $f(x) = a_1^2x_1^2 + \cdots + a_n^2x_n^2, a_1, \cdots , a_n > 0$.

For a fixed $k > 0$ and a small $h > 0$, consider the tangent hyperplane Φ of M_{k+h} at a point $v \in M_{k+h}$ which is parallel to the tangent hyperplane Ψ of M_k at $p \in M_k$. The two points p and v of tangency are related by

$$v = \frac{\sqrt{k+h}}{\sqrt{k}}p = \frac{\sqrt{k+h}}{\sqrt{k}}(p_1, \cdots , p_n, \sqrt{r^2 + k}), r^2 = a_1^2p_1^2 + \cdots + a_n^2p_n^2. \tag{4.1}$$

The tangent hyperplane Φ of M_{k+h} at $v \in M_{k+h}$ is given by

$$\Phi : z = \frac{1}{\sqrt{r^2 + k}}\{a_1^2p_1x_1 + \cdots + a_n^2p_nx_n + \sqrt{k(k + h)}\}. \tag{4.2}$$

The linear transformation $y_1 = a_1x_1, \cdots , y_n = a_nx_n, z = z$ transforms M_k to $M_k' : z^2 = |y|^2 + k$, $p = (p_1, \cdots , p_n, \sqrt{r^2 + k})$ to $q = (a_1p_1, \cdots , a_np_n, \sqrt{r^2 + k})$, and Φ to the hyperplane Φ' defined by

$$\Phi' : z = \frac{1}{\sqrt{|q|^2 + k}}\{\langle q, y\rangle + \sqrt{k(k + h)}\}. \tag{4.3}$$

Hence the n-dimensional surface area $S_p^*(k, h)$ of the region of M_k between the two hyperplanes Φ and Ψ is given by

$$S_p^*(k, h) = \frac{1}{a}\int_{D_q(k,h)} \frac{\{(a_1^2 + 1)y_1^2 + \cdots + (a_n^2 + 1)y_n^2 + k\}^{1/2}}{\sqrt{|y|^2 + k}}dy, \tag{4.4}$$

where $a = a_1 \cdots a_n$ and

$$D_q(k, h) : (|q|^2 + k)(|y|^2 + k) \leq (\langle q, y\rangle + \sqrt{k(k + h)})^2. \tag{4.5}$$

By assumption, $S^*_p(k,h)/|\nabla g(p)| = \eta_k(h)$ is independent of p. Since we have

$$|\nabla g(p)|^2 = 4((a_1^2+1)q_1^2 + \cdots + (a_n^2+1)q_n^2 + k),$$

we see that

$$\int_{D_q(k,h)} H(y)dy = 2a\sqrt{|q|^2+k}\,H(q)\eta_k(h), \tag{4.6}$$

where we denote

$$H(y) = \frac{\{(a_1^2+1)y_1^2 + \cdots + (a_n^2+1)y_n^2 + k\}^{1/2}}{\sqrt{|y|^2+k}}. \tag{4.7}$$

It is straightforward to show that $D_q(k,h)$ is an ellipsoid centered at $\sqrt{(k+h)/k}q$ and its canonical form is given by

$$\frac{ky_1^2}{h(|q|^2+k)} + \frac{y_2^2+\cdots+y_n^2}{h} \le 1. \tag{4.8}$$

This shows that the volume of $D_q(k,h)$ is given by

$$V(D_q(k,h)) = \frac{1}{\sqrt{k}}(\sqrt{h})^n\sqrt{|q|^2+k}\,\omega_n. \tag{4.9}$$

Let's denote by $\theta_k(h)$ the function defined by

$$\theta_k(h) = \frac{2a\sqrt{k}}{\omega_n(\sqrt{h})^n}\eta_k(h). \tag{4.10}$$

Then, it follows from (4.6) and (4.9) that $H(y)$ satisfies

$$\frac{1}{V(D_q(k,h))}\int_{D_q(k,h)} H(y)dy = H(q)\theta_k(h). \tag{4.11}$$

When $q=0$, $H(0)=1$ and $D_0(k,h)$ is the ball $B_0(\sqrt{h})$ of radius $\sqrt{h}$ centered at $y=0$. Hence, (4.11) implies that for any positive numbers k and h

$$\theta_k(h) = \frac{1}{V(B_0(\sqrt{h}))}\int_{B_0(\sqrt{h})} H(y)dy > 1. \tag{4.12}$$

If we let $a_1 = \max\{a_i | i = 1,2,\cdots,n\}$, then we have from (4.7)

$$\begin{aligned} &1 = H(0) \le H(y) < \sqrt{a_1^2+1}, \\ &\lim_{y_1\to\infty} H(y_1,0,\cdots,0) = \sqrt{a_1^2+1}. \end{aligned} \tag{4.13}$$

Thus, the left hand side of (4.11) is less than $\sqrt{a_1^2+1}$ for any positive numbers k, h and $q \in \mathbb{R}^n$. But, since $\theta_k(h) > 1$, for $q = (q_1, 0, \cdots, 0)$ with sufficiently large q_1, the right hand side of (4.11) is greater than $\sqrt{a_1^2+1}$. This contradiction completes the proof of Theorem 5.

5. Elliptic paraboloids

In this section, we prove Theorem 6.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a real number $\alpha \in R$ with $\alpha \neq 0, 2$, let's consider the function g defined by $g(x, z) = z^\alpha - f(x)$. We suppose that the level hypersurfaces $M_k, k \in S_g$ of g in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex.

Suppose that the function g satisfies the condition 1) or 2) in Theorem 6. Then, it follows from Lemma 8 or Lemma 9 that g satisfies the condition 3) in Theorem 6.

First, we show that the condition 3) in Theorem 6 implies 4) as follows. As in the proof of Theorem 2 in Section 3, we can show that if $\alpha \neq 0, 1, 2$, then (3.12) leads to a contradiction. Since $\alpha \neq 0, 2$, the remaining case is for $\alpha = 1$. In this case, it follows from (3.4) that

$$\det(f_{ij}(x)) = c(k). \tag{5.1}$$

Hence we see that $\det(f_{ij})$ is a positive constant c with $c(k) = c$. Thus, $f(x)$ is a quadratic polynomial given by ([1], [4])

$$f(x_1, \cdots, x_n) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, \quad a_1, \cdots, a_n > 0. \tag{5.2}$$

Conversely, suppose that the level hypersurfaces are given by $M_k : z = f(x) + k, z > 0$ with $k > 0$, where $f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, a_1, \cdots, a_n > 0$. In this case, we have $R_g = S_g = R$ and $I_k = (k, \infty)$. From the proof of Theorem 5 in [3], we get

$$V_p^*(k, h) = \gamma_n h^{(n+2)/2} \quad \text{and} \quad A_p^*(k, h) = \frac{n+2}{2} \gamma_n |\nabla g(p)| h^{n/2}, \tag{5.3}$$

where

$$\gamma_n = \frac{2\sigma_{n-1}}{n(n+2)a_1 a_2 \cdots a_n} \tag{5.4}$$

and σ_{n-1} denotes the surface area of the $(n-1)$-dimensional unit sphere. It also follows from (3.3) with $\alpha = 1$ that

$$K(p)|\nabla g(p)|^{n+2} = 2^n a_1^2 a_2^2 \cdots a_n^2. \tag{5.5}$$

This completes the proof of Theorem 6.

Corollary 10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$, let's denote by g the function defined by $g(x, z) = z^\alpha - f(x)$. Suppose that $R_g = R$ and the level hypersurfaces M_k of g in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex for all $k \in R$. Then the following are equivalent.

1) The function g satisfies Condition (V^*).
2) The function g satisfies Condition (A^*).
3) $K(p)|\nabla g(p)|^{n+2} = c(k)$ is constant on each M_k.
4) For some positive constants $a_1, \cdots, a_n$,

$$g(x, z) = z - (a_1^2 x_1^2 + \cdots + a_n^2 x_n^2).$$

Proof. Suppose that the function g satisfies one of the conditions 1), 2) and 3). Then as above, we have $\alpha = 1$ or $\alpha = 2$. In case $\alpha = 2$, $c(k)$ is a nonconstant linear function in k. This contradicts to the positivity of $c(k)$. Hence we have $\alpha = 1$. Thus, Theorem 6 completes the proof. □

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